

LINEAR PARTIAL DIFFERENTIAL EQUATIONS AND REAL ANALYTIC
APPROXIMATIONS OF ROUGH FUNCTIONS

By

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science

in

Mathematics

University of Alaska Fairbanks

August 2017

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Abstract

Many common approximation methods exist such as linear or polynomial interpolation, splines, Taylor series, or generalized Fourier series. Unfortunately, many of these approximations are not analytic functions on the entire real line, and those that are diverge at infinity and therefore are only valid on a closed interval or for compactly supported functions.

Our method takes advantage of the smoothing properties of certain linear partial differential equations to obtain an approximation which is real analytic, converges to the function on the entire real line, and yields particular conservation laws. This approximation method applies to any L_2 function on the real line which may have some rough behavior such as discontinuities or points of nondifferentiability. For comparison, we consider the well-known Fourier-Hermite series approximation. Finally, for some example functions the approximations are found and plotted numerically.

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Acknowledgments

First and foremost I would like to express my deepest gratitude to Dr. Alexei Rybkin. As my graduate committee chair, he was the inspiration for this project and a fountain of knowledge on the subject. Without his patient guidance and support this project would never have come to fruition.

My thanks go to Dr. Sergei Avdonin, Dr. Jill Faudree, and the rest of the mathematics faculty for their assistance and instruction over the many years I have attended the University of Alaska Fairbanks. In particular, many thanks to Dr. John Rhodes for his caring support and for being in my corner when I was in need.

Last but certainly not least, thanks go to my wife Andrea, who has been with me on this journey since the very beginning and has supported me every step of the way. I'd like to thank my mother, my father, and my sister Kimberly, who have never wavered in their belief of me over these long years. Without the love and support of my family, this would not have been possible.

Chapter 1

Introduction

There are several common methods for approximating a real-valued function. Many are familiar with linear or polynomial regression analysis, which is the process of finding a line or curve which best describes a set of discrete data points. The precise meaning of best fit may change depending on the situation, but a common measure is the least squares, the curve which minimizes the sum of the squares of the difference between the actual data point and the corresponding point on the curve.

When making the leap from finite data sets to real-valued functions, approximation methods become more complicated. If a function f is N -times differentiable at a point x_0 , Taylor's theorem gives an approximation for f within a neighborhood of x_0 in terms of the Taylor polynomial [CC12],

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

where $n \leq N$. One could say that this is a local result, as it requires information about the function at the point x_0 , and the approximation may only be accurate in a small neighborhood about x_0 . At points outside this neighborhood, the Taylor polynomial may differ significantly from the function in question. Typically, increasing the number of terms in the polynomial will broaden the neighborhood in which P_n remains a good approximation, but this requires existence of derivatives of high order which is a very restrictive condition.

Fortunately, we can eliminate this differentiability condition with another method. For continuous functions, the Weierstrass approximation theorem has established that on any closed interval, a continuous function can be approximated uniformly by a polynomial. That is, for a function f continuous on an interval $[a, b]$, and for any $\epsilon > 0$ there is a polynomial p which satisfies $|f(x) - p(x)| < \epsilon$ for all points x in the interval $[a, b]$. This is a monumental result, but it is still restricted to a relatively small class of functions. For instance, if the function has a jump discontinuity, it cannot be uniformly approximated by a polynomial.

Additionally, this result only holds on a bounded interval. Since polynomial functions diverge at infinity, there can be no hope of using them to approximate a bounded function on the whole real line.

Fourier series can be used to approximate certain functions that are defined on a bounded interval or are periodic on the whole line. By truncating the Fourier series, a function can be approximated by a finite linear combination of sines and cosines. If the function is continuous, it can be approximated uniformly; if it is an L_2 function, it can be approximated in the L_2 norm, somewhat analogous to the least squares regression. However, this method still has strict conditions on the function in question.

Our intention is to find a method of approximation that relaxes these conditions and applies to a greater class of functions. Specifically, we seek methods of approximation which can be applied to discontinuous or nondifferentiable functions, and those without compact support. To approximate such a real-valued function $f(x)$, we consider a corresponding partial differential equation (PDE) initial value problem (IVP), where the function f is the initial profile for the IVP. Then, the solution $u(x, t)$ of the IVP approximates f for small values of t . The particular choice of IVP imparts u with properties of interest, such as regularity while still maintaining properties from f .

1.1 Solution to IVP by Fourier Transform

In general, the IVP is of the form

$$\begin{cases} u_t &= p(\partial_x)u \\ u|_{t=0} &= F(x), \end{cases} \quad (1.1)$$

where $u = u(x, t)$ and the subscripts denote partial derivatives in that variable. Note that

$$p(\partial_x) = \sum_{k=0}^n a_k \partial_x^k \quad (1.2)$$

is a polynomial in the derivative ∂_x for some constants a_k .

To the partial differential equation in (1.1), we apply the Fourier Transform, [O’N03]

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx,$$

which then gives

$$\mathcal{F}[u_t] = \mathcal{F}[p(\partial_x)u].$$

Since the transform is linear and independent of t , we have

$$\frac{\partial}{\partial t} \mathcal{F}[u] = \sum_{k=0}^n a_k \mathcal{F}[(\partial_x^k)u]. \quad (1.3)$$

Recall, the Fourier transform of a derivative is related to the transform of the function itself,

$$\mathcal{F}[f^{(n)}(x)](\xi) = (i\xi)^n \mathcal{F}[f](\xi)$$

as can be verified by integration by parts [O'N03]. Therefore,

$$\frac{\partial}{\partial t} \mathcal{F}[u] = \sum_{k=0}^n a_k (i\xi)^k \mathcal{F}[u],$$

or in the simpler form,

$$\frac{\partial}{\partial t} \widehat{u} = \sum_{k=0}^n a_k (i\xi)^k \widehat{u}.$$

This differential equation in t has a general solution

$$\widehat{u}(\xi, t) = A \exp \left[\sum_{k=0}^n a_k (i\xi)^k t \right]$$

for some constant A with respect to time. By taking the Fourier Transform of the initial conditions, we find

$$\widehat{u}|_{t=0} = \widehat{F(x)},$$

and hence $A = \widehat{F(x)}$. Thus, we have

$$\widehat{u}(\xi, t) = \widehat{F(x)} \prod_{k=0}^n e^{a_k (i\xi)^k t}.$$

Now, with the inverse Fourier transform \mathcal{F}^{-1} we define

$$K(x, t) = \mathcal{F}^{-1} \left[\prod_{k=0}^n e^{a_k (i\xi)^k t} \right]. \quad (1.4)$$

Then, by taking the Fourier transform we have

$$\widehat{u}(\xi, t) = \widehat{F(x)} \widehat{K(x, t)},$$

and hence

$$u(x, t) = \mathcal{F}^{-1} \left[\widehat{F(x)K(x, t)} \right].$$

Another basic property of the Fourier transform relates multiplication and convolution [O’N03],

$$\widehat{f} \cdot \widehat{g} = \widehat{f * g}.$$

Therefore,

$$u(x, t) = \mathcal{F}^{-1} \left[\widehat{F * K} \right],$$

and hence we conclude

$$u(x, t) = \int_{\mathbb{R}} F(\xi) K(x - \xi, t) d\xi. \tag{1.5}$$

Note that the kernel K closely depends on the coefficients a_k in the differential polynomial (1.2), and hence is unique to the particular form of the IVP (1.1).

Chapter 2

Heat Equation

2.1 Introduction

We first consider the initial value problem for the heat equation on the real line, with a known initial profile $F(x)$,

$$\begin{cases} u_t = u_{xx} \\ u|_{t=0} = F(x) \end{cases} \quad (2.1)$$

where $t > 0$ and $x \in \mathbb{R}$. Physically, $u(x, t)$ describes the temperature at position x and time t in a thin, perfectly insulated material of infinite length. The initial profile F gives the initial temperature distribution along the wire at time $t = 0$.

In general, we assume that any physical system has finite energy, so we require that the initial profile F be an L_2 function on \mathbb{R} ,

$$\int_{-\infty}^{\infty} |F(x)|^2 dx < \infty.$$

For this problem, we are interested in not just any L_2 function, but in discontinuous, non-differentiable, or infinitely supported functions, so we can analyze the smoothing properties of the PDE.

Using the previous results, the IVP (2.1) is solved using the Fourier transform. In this case our differential polynomial (1.2) has only one nonzero term, $a_2 = 1$. Therefore, the kernel (1.4) for this IVP is given by

$$K(x, t) = \mathcal{F}^{-1} \left[e^{-\xi^2 t} \right].$$

Using the Fourier transform pair [O'N03]

$$\mathcal{F} \left[e^{-a^2 x^2} \right] = \frac{\sqrt{\pi}}{a} e^{-\xi^2 / 4a^2} \quad \text{where } a > 0,$$

we can rearrange this and write it in the form

$$\frac{a}{\sqrt{\pi}} e^{-a^2 x^2} = \mathcal{F}^{-1} \left[e^{-\xi^2/4a^2} \right].$$

Since $t > 0$, we make the substitution $a = 1/\sqrt{4t}$, and thus

$$K(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}. \quad (2.2)$$

Therefore, by (1.5) the solution to the heat equation IVP is

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) e^{-(x-\xi)^2/4t} d\xi. \quad (2.3)$$

For $t > 0$, this function is our approximation to the original profile F on the whole real line.

2.2 Analyticity of Solution

Now, we claim that this solution extends to the complex plane, and is an *entire* function (that is, it is analytic on the whole complex plane). For $z \in \mathbb{C}$, where $z = x + iy$,

$$u(z, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) e^{-(z-\xi)^2/4t} d\xi. \quad (2.4)$$

Now, note $(z - \xi)^2 = (x + iy - \xi)^2 = (x - \xi)^2 + 2iy(x - \xi) - y^2$, and hence

$$\begin{aligned} u(z, t) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) e^{-(x-\xi)^2/4t} e^{-2iy(x-\xi)/4t} e^{y^2/4t} d\xi \\ &= \frac{e^{y^2/4t}}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) e^{-(x-\xi)^2/4t} e^{-iy(x-\xi)/2t} d\xi. \end{aligned}$$

Now, we take the modulus of this equation, taking advantage of the triangle inequality for integrals [Car00],

$$|u(z, t)| \leq \frac{e^{y^2/4t}}{\sqrt{4\pi t}} \int_{\mathbb{R}} \left| F(\xi) e^{-(x-\xi)^2/4t} e^{-iy(x-\xi)/2t} \right| d\xi.$$

Note that $|e^{i\alpha}| = 1$ for any real α , and hence

$$|u(z, t)| \leq \frac{e^{y^2/4t}}{\sqrt{4\pi t}} \int_{\mathbb{R}} \left| F(\xi) e^{-(x-\xi)^2/4t} \right| d\xi.$$

Recall that $\|\cdot\|_p$ denotes the L_p norm on \mathbb{R} ,

$$\|f(x)\|_p = \left[\int_{-\infty}^{\infty} |f(x)|^p dx \right]^{1/p}.$$

We make use of the Hölder inequality $\|fg\|_1 \leq \|f\|_2 \|g\|_2$ [Car00],

$$|u(z, t)| \leq \frac{e^{y^2/4t}}{\sqrt{4\pi t}} \|F(\xi)\|_2 \left\| e^{-(x-\xi)^2/4t} \right\|_2.$$

Note that $\|F(\xi)\|_2$ is finite since F is an L_2 function by assumption.

Furthermore, note

$$\begin{aligned} \left\| e^{-(x-\xi)^2/4t} \right\|_2 &= \left[\int_{\mathbb{R}} \left| e^{-(x-\xi)^2/4t} \right|^2 d\xi \right]^{1/2} \\ &= \left[\int_{\mathbb{R}} e^{-(x-\xi)^2/2t} d\xi \right]^{1/2} \\ &= \sqrt[4]{2\pi t}. \end{aligned}$$

Thus, the integral in (2.4) is absolutely convergent, so u extends to a well-defined function in the complex plane.

To show that this approximating function (2.4) is entire, we use Morera's theorem, which states that if f is a continuous function in a simply connected region $U \subseteq \mathbb{C}$, and if

$$\oint_{\gamma} f(z) dz = 0$$

for every simple, closed square path γ in U , then f is analytic on U [AW01, page 413]. First, we prove the following lemma.

Lemma 1. *Let $F(\xi)$ be an L_2 function on \mathbb{R} and $K(\cdot, t)$ on \mathbb{C} defined by*

$$K(z, t) = \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t}.$$

Then for any closed square path S in the complex plane with sides parallel to the axes, on the product space $S \times \mathbb{R}$, the integral

$$\int_{S \times \mathbb{R}} |F(\xi) K(z - \xi, t)| d\xi |dz|$$

is absolutely convergent.

Proof. Without loss of generality, assume S is positively oriented. Starting at the upper right vertex, label the coordinates on the vertices (α_1, β_1) , (α_2, β_1) , (α_2, β_2) , (α_1, β_2) , and the edges Δ_k where $k = 1, \dots, 4$. Since the integrand is non-negative, Tonelli's Theorem states that we can rewrite this as an iterated integral,

$$\begin{aligned} \int_{S \times \mathbb{R}} |F(\xi)K(z - \xi, t)| \, d\xi |dz| &= \int_S \int_{\mathbb{R}} |F(\xi)K(z - \xi, t)| \, d\xi |dz| \\ &= \sum_{k=1}^4 \int_{\Delta_k} \int_{\mathbb{R}} |F(\xi)K(z - \xi, t)| \, d\xi |dz|. \end{aligned}$$

On each Δ_k , we use the Hölder inequality [Car00] to show the integral is absolutely convergent,

$$\begin{aligned} \int_{\Delta_k} \int_{\mathbb{R}} |F(\xi)K(z - \xi, t)| \, d\xi |dz| \\ \leq \left[\int_{\Delta_k} \int_{\mathbb{R}} |F(\xi)|^2 \, d\xi |dz| \right]^{1/2} \left[\int_{\Delta_k} \int_{\mathbb{R}} |K(z - \xi, t)|^2 \, d\xi |dz| \right]^{1/2}. \end{aligned}$$

Since F is an L_2 function, the first integral is finite, so we focus our attention on the second integral. On the top edge Δ_1 , $z = x + i\beta_1$ where x runs from α_1 to α_2 , we have

$$\begin{aligned} \int_{\Delta_1} \int_{\mathbb{R}} |K(z - \xi, t)|^2 \, d\xi |dz| &= \int_{\alpha_1}^{\alpha_2} \int_{\mathbb{R}} \left| \frac{1}{\sqrt{4\pi t}} e^{-(x+i\beta_1-\xi)^2/4t} \right|^2 \, d\xi \, dx \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\alpha_1}^{\alpha_2} \int_{\mathbb{R}} \left| e^{-((x-\xi)^2 + 2i\beta_1(x-\xi) - \beta_1^2)/2t} \right| \, d\xi \, dx \\ &= \frac{e^{\beta_1^2/2t}}{\sqrt{4\pi t}} \int_{\alpha_1}^{\alpha_2} \int_{\mathbb{R}} e^{-(x-\xi)^2/2t} \, d\xi \, dx \end{aligned}$$

Note that the Gaussian integral can be computed directly,

$$\int_{\mathbb{R}} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}}.$$

Hence, with a suitable substitution we find

$$\begin{aligned} \frac{e^{\beta_1^2/2t}}{\sqrt{4\pi t}} \int_{\alpha_1}^{\alpha_2} \int_{\mathbb{R}} e^{-(x-\xi)^2/2t} \, d\xi \, dx &= \frac{e^{\beta_1^2/2t}}{\sqrt{2}} \int_{\alpha_1}^{\alpha_2} \, dx \\ &= \frac{e^{\beta_1^2/2t}}{\sqrt{2}} (\alpha_2 - \alpha_1). \end{aligned}$$

The edge Δ_3 is similar, with $z = x + i\beta_2$, and x runs from α_2 to α_1 . Repeating the same computations as above with the new coordinates we obtain

$$\int_{\Delta_3} \int_{\mathbb{R}} |K(z - \xi, t)|^2 d\xi |dz| = \frac{e^{\beta_2^2/2t}}{\sqrt{2}}(\alpha_1 - \alpha_2).$$

On the edge Δ_2 , $z = \alpha_2 + iy$, where y runs from β_1 to β_2 , we have

$$\begin{aligned} \int_{\Delta_2} \int_{\mathbb{R}} |K(z - \xi, t)|^2 d\xi |dz| &= \int_{\beta_1}^{\beta_2} \int_{\mathbb{R}} \left| \frac{1}{\sqrt{4\pi t}} e^{-(\alpha_2 + iy - \xi)^2/4t} \right|^2 d\xi dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\beta_1}^{\beta_2} \int_{\mathbb{R}} \left| e^{-((\alpha_2 - \xi)^2 + 2iy(\alpha_2 - \xi) - y^2)/2t} \right| d\xi dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\beta_1}^{\beta_2} e^{y^2/2t} \int_{\mathbb{R}} e^{-(\alpha_2 - \xi)^2/2t} d\xi dy \\ &= \frac{1}{\sqrt{2}} \int_{\beta_1}^{\beta_2} e^{y^2/2t} dy \end{aligned}$$

again by use of the Gaussian integral. We estimate the integral by taking the maximum value of the exponential, which is finite on the closed path,

$$\frac{1}{\sqrt{2}} \int_{\beta_1}^{\beta_2} e^{y^2/2t} dy \leq \frac{e^{\beta_1^2/2t}}{\sqrt{2}}(\beta_1 - \beta_2)$$

Finally, on the edge Δ_4 , $z = \alpha_1 + iy$, where y runs from β_2 to β_1 . The computation is similar to the edge Δ_2 , and we find

$$\int_{\Delta_4} \int_{\mathbb{R}} |K(z - \xi, t)|^2 d\xi dz \leq \frac{e^{\beta_1^2/2t}}{\sqrt{2}}(\beta_1 - \beta_2).$$

Since the integral on every edge Δ_k is finite, we conclude that

$$\int_{S \times \mathbb{R}} |F(\xi)K(z - \xi, t)| d\xi |dz|$$

is finite, as desired. □

Theorem 2. *If F is an L_2 function on \mathbb{R} , and*

$$K(z, t) = \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t},$$

then

$$f(z) = \int_{\mathbb{R}} F(\xi) K(z - \xi, t) d\xi$$

is an entire function.

Proof. Let S be a closed square path in the complex plane with sides parallel to the axes. By the Lemma above,

$$\int_{S \times \mathbb{R}} |F(\xi) K(z - \xi, t)| d\xi |dz|$$

is finite so we can interchange the order of integration. Hence,

$$\begin{aligned} \int_S f(z) dz &= \int_S \int_{\mathbb{R}} F(\xi) K(z - \xi, t) d\xi dz \\ &= \int_{\mathbb{R}} \int_S F(\xi) K(z - \xi, t) dz d\xi \\ &= \int_{\mathbb{R}} \left[\int_S K(z - \xi, t) dz \right] F(\xi) d\xi. \end{aligned}$$

Since K is an entire function, the integral over the closed loop S vanishes and hence

$$\int_S f(z) dz = 0.$$

Since S was an arbitrary square in \mathbb{C} , by Morera's Theorem [AW01, page 413], $f(z)$ is analytic on the whole complex plane. \square

2.3 Convergence

Thus, we've shown that for an initial profile $F \in L_2(\mathbb{R})$, the approximating function (2.4) extends to an entire function on the complex plane. Now, we would like to show that the approximating function $u(x, t)$ converges to the initial profile $F(x)$ as $t \rightarrow 0$. To do this, we first define a sequence of functions

$$\phi_t(x) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}.$$

Theorem 3. *The sequence $\{\phi_t\}_{t>0}$ is a δ -sequence.*

Proof. Note that for all $t > 0$, $\phi_t(x) \geq 0$. Also,

$$\int_{\mathbb{R}} \phi_t(x) dx = \int_{\mathbb{R}} \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} = \frac{\sqrt{4\pi t}}{\sqrt{4\pi t}} = 1.$$

Now, we need to show that the sequence converges uniformly to zero outside of any neighborhood of zero. Let $\epsilon > 0$. Then, for any point in $\mathbb{R} \setminus (-\epsilon, \epsilon)$,

$$\|\phi_t(x)\|_\infty = \max_{x \in \mathbb{R} \setminus (-\epsilon, \epsilon)} \left| \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \right| = \frac{e^{-\epsilon^2/4t}}{\sqrt{4\pi t}}.$$

Hence, for all $x \in \mathbb{R} \setminus (-\epsilon, \epsilon)$,

$$\lim_{t \rightarrow 0} \|\phi_t(x)\|_\infty = \frac{1}{2\sqrt{\pi}} \lim_{t \rightarrow 0} \frac{e^{-\epsilon^2/4t}}{\sqrt{t}} = 0.$$

Thus, the sequence $\{\phi_t\}$ is a δ -sequence. □

We conclude,

$$\lim_{t \rightarrow 0} u(x, t) = \lim_{t \rightarrow 0} \int_{\mathbb{R}} F(\xi) \frac{1}{\sqrt{4\pi t}} e^{-(x-\xi)^2/4t} d\xi = \lim_{t \rightarrow 0} \int_{\mathbb{R}} F(\xi) \phi_t(x - \xi) d\xi = F(x).$$

2.4 Conserved Quantities

Now, we claim that if the initial profile is non-negative, the total area is preserved,

$$\int_{\mathbb{R}} u(x, t) dx = \int_{\mathbb{R}} F(x) dx.$$

Note that since the integrand is non-negative, we can change the order of integration and hence,

$$\begin{aligned} \int_{\mathbb{R}} u(x, t) dx &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) e^{-(x-\xi)^2/4t} d\xi dx \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) \int_{\mathbb{R}} e^{-(x-\xi)^2/4t} dx d\xi \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) (\sqrt{4\pi t}) d\xi \\ &= \int_{\mathbb{R}} F(\xi) d\xi. \end{aligned}$$

This is an expected result, as the heat equation preserves total area. Consider $u(x, t)$ to be some $L_2(\mathbb{R})$ function which is a solution to the heat equation for $x \in \mathbb{R}$ and $t > 0$. Note that by necessity,

$$\lim_{|x| \rightarrow \infty} u = \lim_{|x| \rightarrow \infty} u_x = 0.$$

Since $u_t = u_{xx}$, we integrate this equation over the entire real line,

$$\int_{\mathbb{R}} u_t dx = \int_{\mathbb{R}} u_{xx} dx.$$

Since t is independent of x ,

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} u dx = \int_{\mathbb{R}} \frac{\partial}{\partial x} u_x dx = u_x \Big|_{-\infty}^{\infty} = 0.$$

That is, $\int_{\mathbb{R}} u dx$ is constant in time, as claimed.

However, we can similarly show that the integral of the square is *not* preserved. For $u_t = u_{xx}$, we multiply this equation by u , then integrate over the whole real line,

$$\int_{\mathbb{R}} u \cdot u_t dx = \int_{\mathbb{R}} u \cdot u_{xx} dx.$$

Note that $(u^2)_t = 2u \cdot u_t$, and hence

$$\frac{1}{2} \int_{\mathbb{R}} (u^2)_t dx = \int_{\mathbb{R}} u \cdot u_{xx} dx.$$

Now, we integrate by parts to yield

$$\frac{1}{2} \int_{\mathbb{R}} (u^2)_t dx = u \cdot u_x \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} u_x \cdot u_x dx = - \int_{\mathbb{R}} (u_x)^2 dx.$$

That is,

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} u^2 dx = -2 \int_{\mathbb{R}} (u_x)^2 dx.$$

But since $(u_x)^2$ is nonnegative, $\int_{\mathbb{R}} (u_x)^2 = 0$ only if u_x is identically zero. This would imply u is trivial, which we disregard. Therefore, $\int_{\mathbb{R}} (u_x)^2 > 0$, and hence $\frac{\partial}{\partial t} \int_{\mathbb{R}} u^2 \neq 0$. That is, $\int_{\mathbb{R}} u^2$ has an explicit time dependence and therefore cannot be conserved.

2.5 Example Profiles

For the preceding analysis, we only assumed that F was an L_2 function on \mathbb{R} . What we are truly interested in are functions which are discontinuous, nondifferentiable, and may be supported on the whole real line. We will give a couple example functions, and find the corresponding approximating function via the heat equation IVP. Then, the approximating function is plotted to show convergence to the original function.

2.5.1 Box Function

First we consider the box function

$$F(x) = \begin{cases} h & \text{if } -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

which is plotted below in Figure 2.1. This function is of interest because it has two discontinuities, which precludes approximation by Taylor series or Weierstrass approximation theorem at those discontinuities.

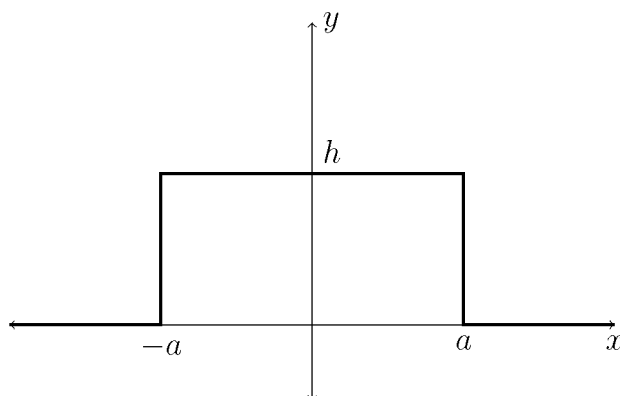


Figure 2.1: The box function of height h with discontinuities at $x = -a, a$. This function is approximated by the solution to the heat equation IVP.

Using the results above, we compute the solution directly,

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) e^{-(x-\xi)^2/4t} d\xi \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-a}^a h e^{-(x-\xi)^2/4t} d\xi. \end{aligned}$$

Now, we make the substitution $u = \frac{\xi - x}{\sqrt{4t}}$, which gives $du = \frac{d\xi}{\sqrt{4t}}$, and also changes the limits of integration,

$$\begin{aligned}
u(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{\frac{-a-x}{\sqrt{4t}}}^{\frac{a-x}{\sqrt{4t}}} h e^{-u^2} \sqrt{4t} du \\
&= \frac{h}{2} \left[\frac{2}{\sqrt{\pi}} \int_0^{\frac{a-x}{\sqrt{4t}}} e^{-u^2} du - \frac{2}{\sqrt{\pi}} \int_0^{\frac{-a-x}{\sqrt{4t}}} e^{-u^2} du \right] \\
&= \frac{h}{2} \left[\operatorname{erf} \left(\frac{a-x}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{-a-x}{\sqrt{4t}} \right) \right] \\
&= \frac{h}{2} \left[\operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) \right]
\end{aligned}$$

where the error function erf is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

Using $a = h = 1$ for simplicity, a representation of this solution is given below in Figure 2.2, in 4 sub-plots for decreasing t values $t = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$. Note that we can clearly see the smooth function taking on the shape of the original discontinuous box profile.

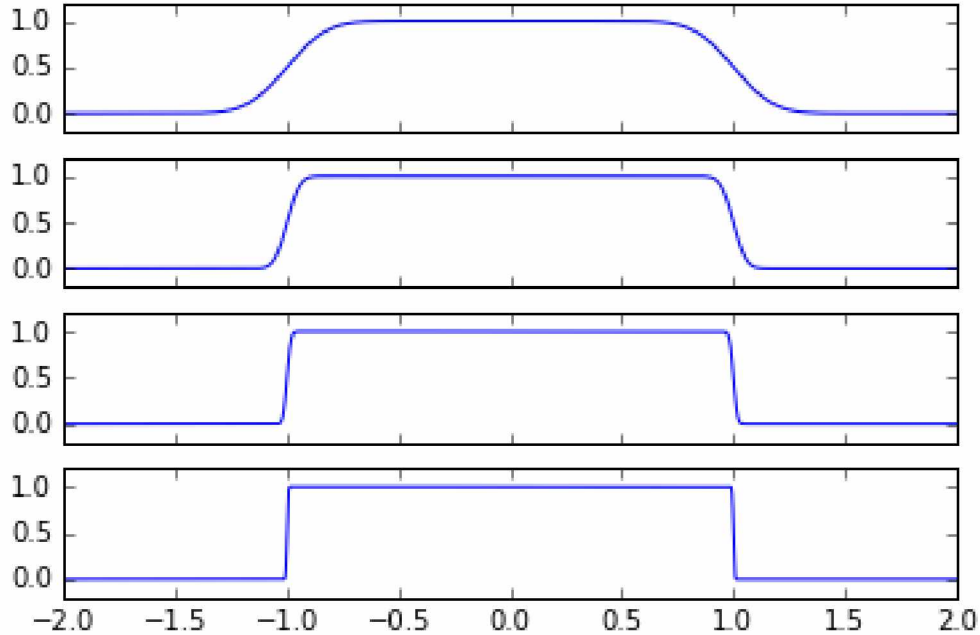


Figure 2.2: Approximating function $u(x, t)$ for the box function, found via the solution to the heat equation IVP. The four plots shown are calculated as t decreases towards zero.

To demonstrate the accuracy of the approximation, Figure 2.3 shows a comparison of the approximating function $u(x, t)$ with $t = 10^{-5}$ to a plot of the original box function. The approximating function is plotted as the continuous blue line, and the box function with dots. Note that even with this moderately small value for t , the approximating function is nearly indistinguishable from the original function.

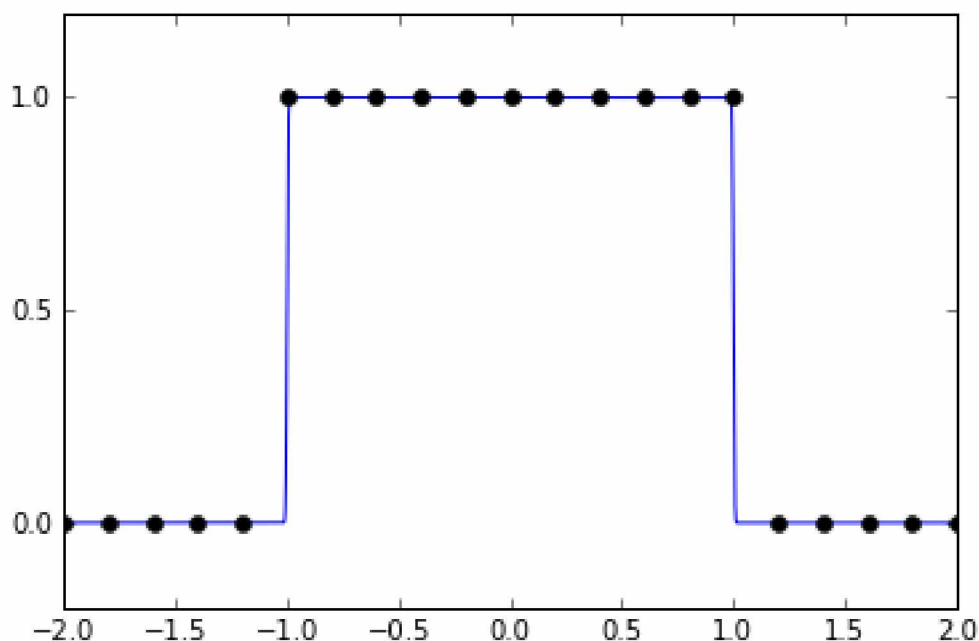


Figure 2.3: The approximating function $u(x, t)$ from the heat equation IVP with $t = 10^{-5}$ in blue, plotted against the original box function, plotted with dots.

The question naturally arises, how well does this smooth function approximate the initial profile for small t ? We'd like to estimate the pointwise error as a function of time,

$$\epsilon(x, t) = |F(x) - u(x, t)|.$$

First, we consider the error at the jump discontinuities, where $|x| = a$. For $x = a$ we note that erf vanishes at zero and hence,

$$\epsilon(x, t) = \left| h - \frac{h}{2} \text{erf} \left(\frac{2a}{\sqrt{4t}} \right) \right|.$$

For the case $x = -a$, we get the same result since the error function is odd. Note that as $t \rightarrow 0$, $\text{erf} \left(\frac{2a}{\sqrt{4t}} \right) \rightarrow 1$, and hence $\epsilon(\pm a, t) \approx h/2$ for very small t . Note that this result is not unexpected; the solution u at the jump discontinuity converges to the average of the left-

and right-hand limits, just like we would see with a Fourier series. The great advantage here is we do not see any Gibbs Phenomenon, as we can verify by plotting the solution u , and by calculating the error in the other regions.

Now, to calculate the error in the region $(-a, a)$, we note that since the solution u is even, it is sufficient to consider only $[0, a)$. Then,

$$\epsilon(x, t) = \left| h - \frac{h}{2} \left[\operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) \right] \right|.$$

Since the error function lies in the interval $[-1, 1]$,

$$\operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) \leq 2,$$

and hence

$$\epsilon(x, t) = h - \frac{h}{2} \left[\operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) \right].$$

Now if we simplify and use the fact that since the error function is increasing,

$$\operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) < \operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right),$$

we have

$$\begin{aligned} 1 - \frac{\epsilon}{h} &= \frac{1}{2} \left[\operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right) - \left(\frac{x-a}{\sqrt{4t}} \right) \right] \\ &\leq \frac{1}{2} \left[2 \operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right) \right]. \end{aligned}$$

Then, using the fact that the inverse error function is also increasing, we find

$$t \leq \frac{(x+a)^2}{4 \left(\operatorname{erf}^{-1} \left(1 - \frac{\epsilon}{h} \right) \right)^2}.$$

Note that this bound is best when $x = 0$, and hence

$$t \leq \frac{a^2}{4 \left(\operatorname{erf}^{-1} \left(1 - \frac{\epsilon}{h} \right) \right)^2}.$$

2.5.2 Tent Function

Now we consider the initial profile of a triangular tent function

$$F(x) = \begin{cases} \frac{h}{a}x + h & \text{if } -a \leq x \leq 0 \\ \frac{-h}{a}x + h & \text{if } 0 < x \leq a \\ 0 & \text{otherwise} \end{cases}$$

which is illustrated below in Figure 2.4. While this function is now continuous, note that it is nondifferentiable at three points.

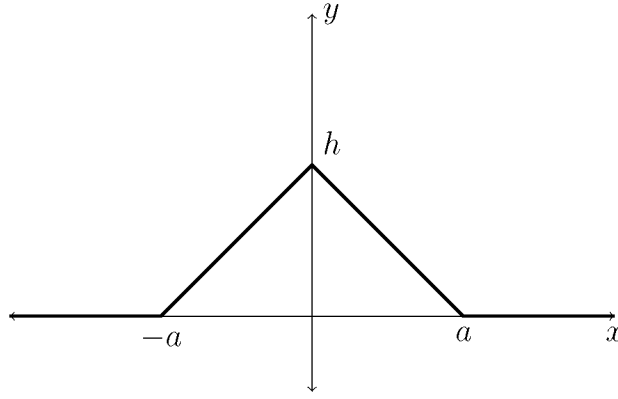


Figure 2.4: The tent function of height h . Note that this function is nondifferentiable at three points.

We compute the solution directly,

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) e^{-(x-\xi)^2/4t} d\xi \\ &= \frac{1}{\sqrt{4\pi t}} \left[\int_{-a}^0 \left(\frac{h}{a}\xi + h \right) e^{-(x-\xi)^2/4t} d\xi + \int_0^a \left(\frac{-h}{a}\xi + h \right) e^{-(x-\xi)^2/4t} d\xi \right] \\ &= \frac{1}{\sqrt{4\pi t}} \left[\frac{h}{a} \int_{-a}^0 \xi e^{-(x-\xi)^2/4t} d\xi + h \int_{-a}^0 e^{-(x-\xi)^2/4t} d\xi \right. \\ &\quad \left. + \frac{-h}{a} \int_0^a \xi e^{-(x-\xi)^2/4t} d\xi + h \int_0^a e^{-(x-\xi)^2/4t} d\xi \right]. \end{aligned}$$

Now, to compute these integrals, first we note that by a suitable substitution we find

$$\int_0^a e^{-(x-\xi)^2/4t} d\xi = -\sqrt{\pi t} \left[\operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x}{\sqrt{4t}} \right) \right].$$

Similarly, we note that

$$\int_0^a \xi e^{-(x-\xi)^2/4t} d\xi = \int_0^a x e^{-(x-\xi)^2/4t} d\xi - \int_0^a (x-\xi) e^{-(x-\xi)^2/4t} d\xi.$$

The first integral can be computed using the identity above. In the second integral, we make the substitution $u = -(x-\xi)^2/4t$, which gives $du = (x-\xi)/2t d\xi$, and hence

$$\begin{aligned} \int_0^a (x-\xi) e^{-(x-\xi)^2/4t} d\xi &= \int_{-x^2/4t}^{-(x-a)^2/4t} e^u (2t) du \\ &= 2t \left[e^{-(x-a)^2/4t} - e^{-x^2/4t} \right], \end{aligned}$$

and hence

$$\int_0^a \xi e^{-(x-\xi)^2/4t} d\xi = -x\sqrt{\pi t} \left[\operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x}{\sqrt{4t}} \right) \right] - 2t \left[e^{-(x-a)^2/4t} - e^{-x^2/4t} \right].$$

Using these results, we find

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi t}} \left[-\frac{h}{a} \int_0^{-a} \xi e^{-(x-\xi)^2/4t} d\xi - h \int_0^{-a} e^{-(x-\xi)^2/4t} d\xi \right. \\ &\quad \left. - \frac{h}{a} \int_0^a \xi e^{-(x-\xi)^2/4t} d\xi + h \int_0^a e^{-(x-\xi)^2/4t} d\xi \right] \\ &= \frac{1}{\sqrt{4\pi t}} \left\{ -\frac{h}{a} \left[-x\sqrt{\pi t} \left[\operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x}{\sqrt{4t}} \right) \right] \right. \right. \\ &\quad \left. \left. - 2t \left[e^{-(x+a)^2/4t} - e^{-x^2/4t} \right] \right] \right. \\ &\quad \left. + h\sqrt{\pi t} \left[\operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x}{\sqrt{4t}} \right) \right] \right. \\ &\quad \left. + \frac{-h}{a} \left[-x\sqrt{\pi t} \left[\operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x}{\sqrt{4t}} \right) \right] \right. \right. \\ &\quad \left. \left. - 2t \left[e^{-(x-a)^2/4t} - e^{-x^2/4t} \right] \right] \right. \\ &\quad \left. - h\sqrt{\pi t} \left[\operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x}{\sqrt{4t}} \right) \right] \right\}, \end{aligned}$$

and when we collect like terms we find

$$\begin{aligned}
u(x, t) = & \frac{hx}{2a} \left[\operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right) - 2\operatorname{erf} \left(\frac{x}{\sqrt{4t}} \right) + \operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) \right] \\
& + \frac{ht}{a\sqrt{\pi t}} \left[e^{-(x-a)^2/4t} - 2e^{-x^2/4t} + e^{-(x+a)^2/4t} \right] \\
& + \frac{h}{2} \left[\operatorname{erf} \left(\frac{x+a}{\sqrt{4t}} \right) - \operatorname{erf} \left(\frac{x-a}{\sqrt{4t}} \right) \right].
\end{aligned}$$

Again, we set $a = h = 1$ for simplicity, and we plot the approximating function $u(x, t)$ for decreasing values of t . This is plotted in Figure 2.5 using the values $t = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$.

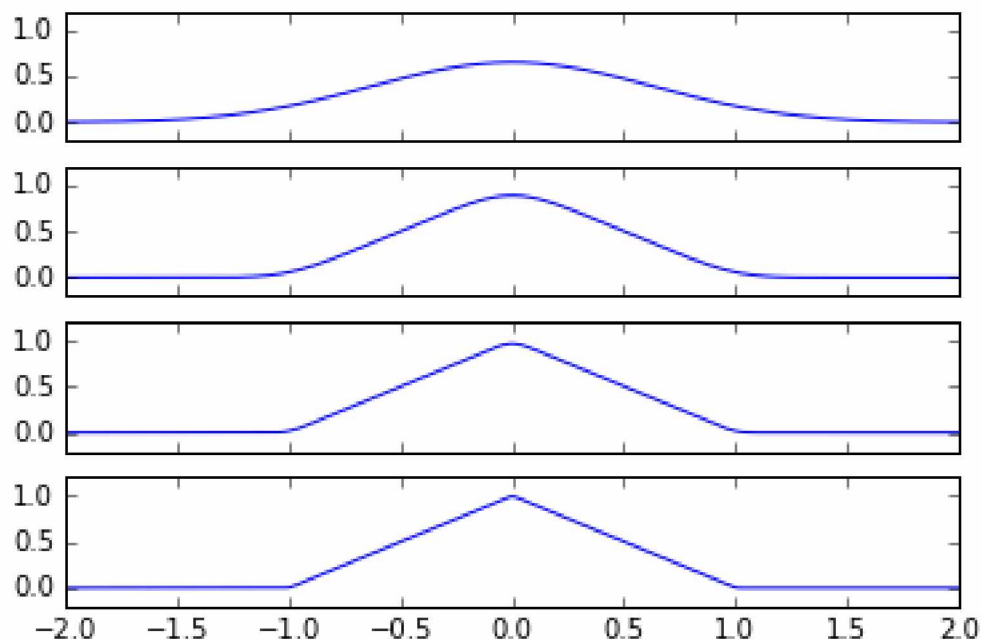


Figure 2.5: Approximation of the triangular tent function by solution of the heat equation IVP.

2.5.3 Cusp Function

Consider the profile

$$F(x) = e^{-\alpha|x|},$$

for some $\alpha > 0$, which is plotted below in Figure 2.6.

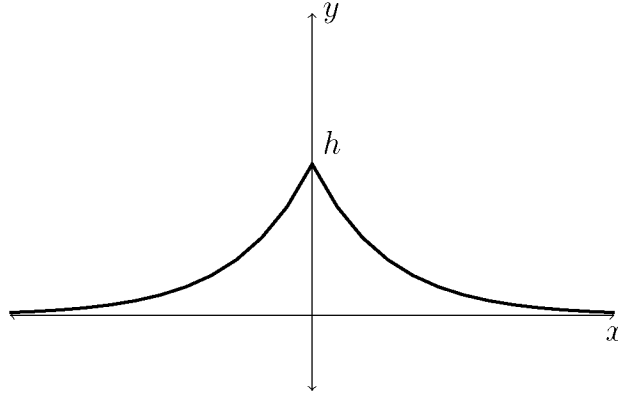


Figure 2.6: The cusp-shaped function, supported on the whole line, with a point of nondifferentiability.

Note that this is an L_2 function on \mathbb{R} , and though it is continuous, it is not differentiable at $x = 0$. We compute the solution,

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\alpha|\xi|} e^{-(x-\xi)^2/4t} d\xi \\ &= \frac{1}{\sqrt{4\pi t}} \left[- \int_0^{-\infty} e^{-(x-\xi)^2/4t + \alpha\xi} d\xi + \int_0^{\infty} e^{-(x-\xi)^2/4t - \alpha\xi} d\xi \right]. \end{aligned}$$

We compute these two integrals using the identity

$$\int_0^{\infty} \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta} \exp(\beta\gamma^2) [1 - \operatorname{erf}(\gamma\sqrt{\beta})]$$

where $\operatorname{Re} \beta > 0$ [GR07, Table 3.322.2]. For the first integral, we make the substitution $z = -\xi$,

$$\begin{aligned}
-\int_0^{-\infty} e^{-\frac{(x-\xi)^2}{4t} + \alpha\xi} d\xi &= \int_0^{\infty} e^{-\frac{(x+z)^2}{4t} - \alpha z} dz \\
&= \int_0^{\infty} e^{-\frac{x^2}{4t} - \frac{xz}{2t} - \frac{z^2}{4t} - \alpha z} dz \\
&= e^{-x^2/4t} \int_0^{\infty} e^{-\frac{z^2}{4t} - (\alpha + x/2t)z} dz \\
&= e^{-x^2/4t} \sqrt{\pi t} e^{t(\alpha + x/2t)^2} \left[1 - \operatorname{erf} \left(\sqrt{t} (\alpha + x/2t) \right) \right] \\
&= \sqrt{\pi t} \exp(\alpha^2 t + \alpha x) \operatorname{erfc} \left(\alpha \sqrt{t} + \frac{x}{2\sqrt{t}} \right),
\end{aligned}$$

where $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$. Similarly for the second integral,

$$\int_0^{\infty} e^{-\frac{(x-\xi)^2}{4t} - \alpha\xi} d\xi = \sqrt{\pi t} \exp(\alpha^2 t - \alpha x) \operatorname{erfc} \left(\alpha \sqrt{t} - \frac{x}{2\sqrt{t}} \right).$$

Thus, we have

$$u(x, t) = \frac{e^{\alpha^2 t}}{2} \left[e^{-\alpha x} \operatorname{erfc} \left(\alpha \sqrt{t} - \frac{x}{2\sqrt{t}} \right) + e^{\alpha x} \operatorname{erfc} \left(\alpha \sqrt{t} + \frac{x}{2\sqrt{t}} \right) \right].$$

We plot the solution below in Figure 2.7, using $\alpha = 1$, for $t = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$.

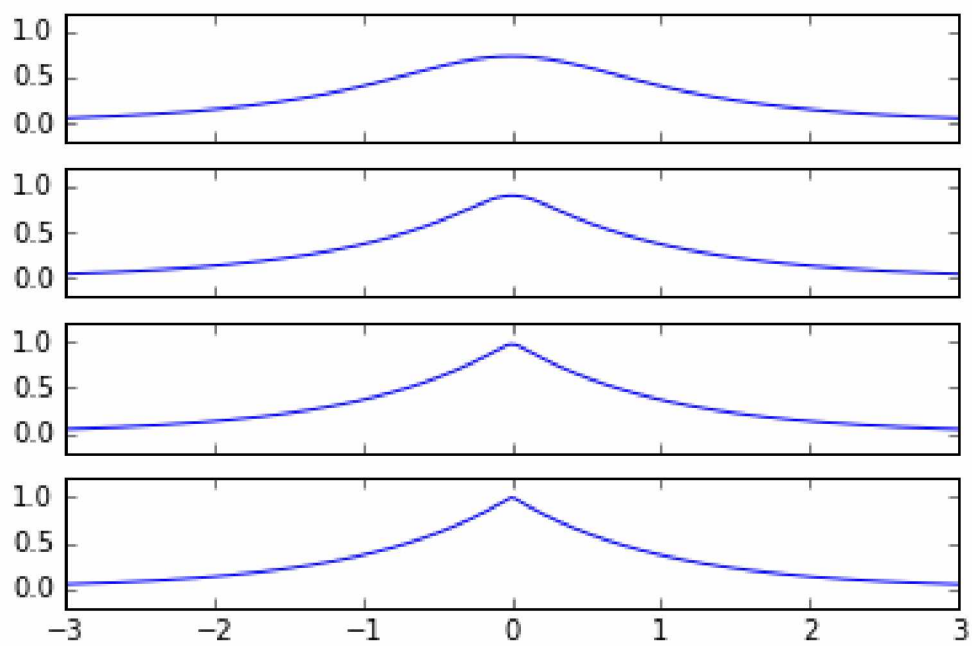


Figure 2.7: Approximation of the cusp function by the heat equation IVP.

Chapter 3

Airy Equation

3.1 Introduction

Now, consider the Airy equation on the real line, or sometimes called the linearized Korteweg-deVries Equation,

$$\begin{cases} u_t + u_{xxx} = 0 \\ u|_{t=0} = F(x), \quad x \in \mathbb{R} \end{cases} \quad (3.1)$$

where the initial profile F is an L_2 function on the real line. This dispersive equation appears in fluid dynamics, including the propagation of long waves in shallow water.

Recall, the solution to the general problem using the Fourier transform method is

$$u(x, t) = \int_{\mathbb{R}} F(\xi) K(x - \xi, t) d\xi,$$

where

$$K(x, t) = \mathcal{F}^{-1} \left[\prod_{k=0}^n e^{a_k(i\xi)^k t} \right].$$

In this case, there is only one nonzero term, $a_3 = -1$. Using the definition of the inverse Fourier transform [O’N03],

$$\mathcal{F}^{-1}[\hat{f}(\xi)](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi.$$

Therefore, the kernel K for this equation is given by

$$K(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi^3 t} e^{i\xi x} d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(t\xi^3 + x\xi)} d\xi.$$

Note that the Airy function [RHB06] [VS04] is defined by

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[i \left(\frac{1}{3} s^3 + xs \right) \right] ds, \quad (3.2)$$

and thus by a suitable substitution we find

$$K(x, t) = \frac{1}{\sqrt[3]{3t}} \text{Ai} \left(\frac{x}{\sqrt[3]{3t}} \right).$$

Therefore, the solution to (3.1) is given by

$$u(x, t) = \frac{1}{\sqrt[3]{3t}} \int_{-\infty}^{\infty} F(\xi) \text{Ai} \left(\frac{x - \xi}{\sqrt[3]{3t}} \right) d\xi.$$

3.2 Complex Extension and Analyticity

As we did for the solution to the heat problem, we want to show that the solution u to the Airy problem extends to the complex plane. First, we need to show that the Airy function itself extends to the complex plane. This is a well-known result, but it is included here in the interest of convenience and completeness of exposition. For the Airy function (3.2), we deform the path of integration along the real line to a line $x + ih$ in the complex plane, for some positive constant h . By definition of the improper integral,

$$\begin{aligned} \text{Ai}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\frac{1}{3}s^3 + xs)} ds \\ &= \frac{1}{2\pi} \lim_{N \rightarrow \infty} \int_{-N}^N e^{i(\frac{1}{3}s^3 + xs)} ds \\ &= \frac{1}{2\pi} \lim_{N \rightarrow \infty} \int_{\Gamma_N} e^{i(\frac{1}{3}s^3 + xs)} ds \end{aligned}$$

where Γ_N is the deformation of the real path of integration into the complex plane as shown below in Figure 3.1.

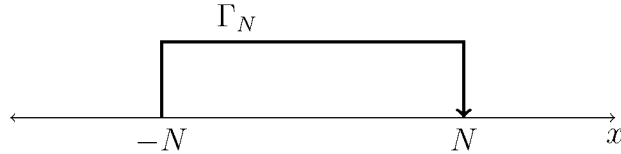


Figure 3.1: Deformation of the path of integration for the Airy function into the complex plane.

For convenience, let us denote this path integral by $I_N(x)$,

$$I_N(x) = \int_{\Gamma_N} e^{i(\frac{1}{3}s^3 + xs)} ds.$$

The path Γ_N has three distinct components which we will consider separately. For convenience, we denote these three integrals by J_1 , J_2 , and J_3 , respectively.

$$I_N(x) = \int_{-N}^{-N+ih} e^{i(\frac{1}{3}s^3+xs)} ds + \int_{-N+ih}^{N+ih} e^{i(\frac{1}{3}s^3+xs)} ds + \int_{N+ih}^N e^{i(\frac{1}{3}s^3+xs)} ds. \quad (3.3)$$

Note that $\text{Ai}(x) - I_N(x)$ goes to zero as $N \rightarrow \infty$. Therefore, we are done if we can show that both J_1 and J_3 vanish as $N \rightarrow \infty$.

For J_1 , we note that $s = -N + i\beta$, where β ranges from 0 to h ,

$$\begin{aligned} J_1 &= \int_{-N}^{-N+ih} e^{i(\frac{1}{3}s^3+xs)} ds \\ &= \int_0^h e^{(-N^2\beta+\beta^3/3-x\beta)+i(-N^3/3+N\beta^2-xN)} i d\beta. \end{aligned}$$

Hence,

$$|J_1| \leq \int_0^h e^{-N^2\beta+\beta^3/3-x\beta} d\beta.$$

Note that the integrand is bounded by $e^{\beta^3/3-x\beta}$, which is finite for fixed x , and hence

$$\lim_{N \rightarrow \infty} \int_0^h e^{-N^2\beta+\beta^3/3-x\beta} d\beta = \int_0^h \lim_{N \rightarrow \infty} e^{-N^2\beta+\beta^3/3-x\beta} d\beta = 0.$$

And thus J_1 vanishes as $N \rightarrow \infty$.

Similarly, for J_3 , we note that $s = N + i\beta$, where β ranges from h to 0,

$$\begin{aligned} J_3 &= \int_{N+ih}^N e^{i(\frac{1}{3}s^3+xs)} ds \\ &= \int_h^0 e^{(-N^2\beta+\beta^3/3-x\beta)+i(N^3/3-N\beta^2+xN)} i d\beta. \end{aligned}$$

Hence,

$$|J_3| \leq \int_h^0 e^{-N^2\beta+\beta^3/3-x\beta} d\beta.$$

As shown for J_1 , this integral, and hence J_3 itself, vanish as $N \rightarrow \infty$.

Thus, we see that since both J_1 and J_3 vanish in the limit, (3.3) reduces to

$$\begin{aligned} \text{Ai}(x) &= \frac{1}{2\pi} \lim_{N \rightarrow \infty} \int_{\Gamma_N} e^{i(\frac{1}{3}s^3+xs)} ds \\ &= \frac{1}{2\pi} \lim_{N \rightarrow \infty} \int_{-N+ih}^{N+ih} e^{i(\frac{1}{3}s^3+xs)} ds, \end{aligned}$$

and thus

$$Ai(x) = \frac{1}{2\pi} \int_{\Gamma} e^{i(\frac{1}{3}s^3 + xs)} ds, \quad (3.4)$$

where Γ is the horizontal line $\alpha + ih$, for real α and fixed $h > 0$.

Remarkably, the integral over the real line in (3.2) is not absolutely convergent, but the integral over Γ in (3.4) is absolutely convergent. Note that on Γ , $s = \alpha + ih$, and hence

$$\begin{aligned} \int_{-N+ih}^{N+ih} \left| e^{i(\frac{1}{3}s^3 + xs)} \right| |ds| &= \int_{-N}^N e^{h^3/3 - hx - h\alpha^2} d\alpha \\ &= e^{\frac{h^3}{3} - hx} \int_{-N}^N e^{-h\alpha^2} d\alpha. \end{aligned}$$

Note that the coefficient $e^{\frac{h^3}{3} - hx}$ is finite for a fixed x , and the integral is also finite. Indeed, this is a truncated Gaussian integral, and hence

$$\begin{aligned} \int_{\Gamma} \left| e^{i(\frac{1}{3}s^3 + xs)} \right| ds &= \lim_{N \rightarrow \infty} \int_{-N+ih}^{N+ih} \left| e^{i(\frac{1}{3}s^3 + xs)} \right| ds \\ &= e^{\frac{h^3}{3} - hx} \int_{-\infty}^{\infty} e^{-h\alpha^2} d\alpha \\ &= e^{\frac{h^3}{3} - hx} \sqrt{\frac{\pi}{h}}, \end{aligned}$$

which is finite for any real x since $h > 0$.

Now, we show that the Airy function is absolutely convergent for any complex argument,

$$Ai(x + iy) = \frac{1}{2\pi} \int_{\Gamma} e^{i(\frac{1}{3}s^3 + (x+iy)s)} ds.$$

On the path Γ , $s = \alpha + ih$, and we expand and simplify the argument,

$$\begin{aligned} \left(\frac{1}{3}(\alpha + ih)^3 + (x + iy)(\alpha + ih) \right) &= \left(-\alpha^2 h + \frac{1}{3}h^3 - hx - \alpha y \right) \\ &\quad + i \left(\frac{1}{3}\alpha^3 - \alpha h^2 + \alpha x - hy \right), \end{aligned}$$

and hence,

$$\begin{aligned}
|\text{Ai}(x + iy)| &\leq \frac{1}{2\pi} \int_{\Gamma} \left| e^{i(\frac{1}{3}s^3 + (x+iy)s)} \right| |ds| \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\alpha^2 h + \frac{1}{3}h^3 - hx - \alpha y} d\alpha \\
&= \frac{1}{2\pi} e^{\frac{1}{3}h^3 - hx} \int_{\mathbb{R}} e^{-\alpha^2 h - \alpha y} d\alpha.
\end{aligned}$$

To integrate, we complete the square in the exponent,

$$\int_{\mathbb{R}} e^{-\alpha^2 h - \alpha y} d\alpha = e^{y^2/h} \int_{\mathbb{R}} e^{-h(\alpha + \frac{y}{2h})^2} = e^{y^2/h} \sqrt{\frac{\pi}{h}}$$

and thus we have the estimate

$$|\text{Ai}(x + iy)| \leq \frac{1}{\sqrt{4\pi h}} e^{h^3/3 + y^2/h} e^{-hx}$$

which is finite for all points $x + iy$, and for any h . We denote this estimate more simply as

$$|\text{Ai}(x + iy)| \leq C(h, y) e^{-hx} \tag{3.5}$$

where $C(h, y)$ is a finite constant for all real y and $h > 0$.

Lemma 4. *If $F(x)$ is bounded on \mathbb{R} and order e^{-kx} as $x \rightarrow \infty$ for some $k > 0$, then for any closed and bounded square contour S in the complex plane, the integral*

$$\int_{S \times \mathbb{R}} \left| F(\xi) \text{Ai} \left(\frac{z - \xi}{\sqrt[3]{3t}} \right) \right| dA$$

is finite.

Proof. Let $z = x + iy$, and we separate the real and imaginary parts of the input,

$$\text{Ai} \left(\frac{z - \xi}{\sqrt[3]{3t}} \right) = \text{Ai} \left(\frac{x - \xi}{\sqrt[3]{3t}} + i \frac{y}{\sqrt[3]{3t}} \right).$$

Then, by (3.5),

$$\begin{aligned}
\int_{S \times \mathbb{R}} \left| F(\xi) \text{Ai} \left(\frac{z - \xi}{\sqrt[3]{3t}} \right) \right| dA &= \int_S \int_{\mathbb{R}} |F(\xi)| \left| \text{Ai} \left(\frac{z - \xi}{\sqrt[3]{3t}} \right) \right| d\xi |dz| \\
&\leq \int_S \int_{\mathbb{R}} |F(\xi)| C \left(h, \frac{y}{\sqrt[3]{3t}} \right) e^{-h \frac{x - \xi}{\sqrt[3]{3t}}} d\xi |dz| \\
&= \int_S C \left(h, \frac{y}{\sqrt[3]{3t}} \right) e^{-h \frac{x}{\sqrt[3]{3t}}} \int_{\mathbb{R}} |F(\xi)| e^{\frac{h}{\sqrt[3]{3t}} \xi} d\xi |dz|.
\end{aligned}$$

Pick x_1 so that $|F(x)| \leq Ae^{-kx}$ for all $x \geq x_1$ where A is a finite constant. Since F is bounded, there exists some constant B such that $|F(x)| \leq B$ for all $x \leq x_1$. Then, if we consider the inner integral only,

$$\begin{aligned}
\int_{\mathbb{R}} |F(\xi)| e^{\frac{h}{\sqrt[3]{3t}} \xi} d\xi &= \int_{-\infty}^{x_1} |F(\xi)| e^{\frac{h}{\sqrt[3]{3t}} \xi} d\xi + \int_{x_1}^{\infty} |F(\xi)| e^{\frac{h}{\sqrt[3]{3t}} \xi} d\xi \\
&\leq \int_{-\infty}^{x_1} B e^{\frac{h}{\sqrt[3]{3t}} \xi} d\xi + \int_{x_1}^{\infty} A e^{-k\xi} e^{\frac{h}{\sqrt[3]{3t}} \xi} d\xi.
\end{aligned}$$

Note that the first integral is finite for all $h > 0$, and the second integral is finite provided $k > h/\sqrt[3]{3t}$.

Thus, we can say that

$$\int_{\mathbb{R}} |F(\xi)| e^{\frac{h}{\sqrt[3]{3t}} \xi} d\xi \leq M$$

for some finite constant M . Therefore,

$$\begin{aligned}
\int_{S \times \mathbb{R}} \left| F(\xi) \text{Ai} \left(\frac{z - \xi}{\sqrt[3]{3t}} \right) \right| dA &\leq M \int_S C \left(h, \frac{y}{\sqrt[3]{3t}} \right) e^{-h \frac{x}{\sqrt[3]{3t}}} |dz| \\
&\leq M \max_{x+iy \in S} \left| C \left(h, \frac{y}{\sqrt[3]{3t}} \right) e^{-h \frac{x}{\sqrt[3]{3t}}} \right| \int_S |dz|,
\end{aligned}$$

which is finite, as claimed. \square

Theorem 5. *If F is bounded on \mathbb{R} and order e^{-kx} as $x \rightarrow \infty$ for some $k > 0$, on \mathbb{R} , and*

$$K(z, t) = \frac{1}{\sqrt[3]{3t}} \text{Ai} \left(\frac{z}{\sqrt[3]{3t}} \right),$$

then

$$f(z) = \int_{\mathbb{R}} F(\xi) K(z - \xi) d\xi$$

is an entire function.

Proof. Let S be a closed square contour in the complex plane. Since the Airy function is an entire function its integral over a closed contour vanishes,

$$\int_S K(z, t) dz = 0.$$

We consider

$$\int_S f(z) dz = \int_S \int_{\mathbb{R}} F(\xi) K(z - \xi) d\xi dz.$$

By Lemma 4 above, this integral is absolutely convergent, and hence by Fubini's Theorem we can interchange the order of integration,

$$\begin{aligned} \int_S f(z) dz &= \int_S \int_{\mathbb{R}} F(\xi) K(z - \xi) d\xi dz \\ &= \int_{\mathbb{R}} \int_S F(\xi) K(z - \xi) dz d\xi \\ &= \int_{\mathbb{R}} \left[\int_S K(z - \xi) dz \right] F(\xi) d\xi \\ &= 0. \end{aligned}$$

Hence by Morera's Theorem [AW01, page 413], $f(z)$ is analytic on the complex plane. \square

3.3 Conserved Quantities

Similar to the solution to the heat equation, the solution $u(x, t)$ to the Airy equation conserves the total area. We note that u and its derivatives must vanish in the limit, in particular,

$$\lim_{|x| \rightarrow \infty} u_{xx} = 0.$$

The Airy equation $u_t = -u_{xxx}$ is integrated over the whole line,

$$\int_{\mathbb{R}} u_t dx = \int_{\mathbb{R}} -u_{xxx} dx.$$

Then, we note,

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} u dx = - \int_{\mathbb{R}} u_{xxx} dx = -u_{xx} \Big|_{-\infty}^{\infty} = 0.$$

That is, $\int_{\mathbb{R}} u dx$ is a constant in time, so it is conserved,

$$\int_{\mathbb{R}} u(x, t) dx = \int_{\mathbb{R}} F(x) dx.$$

Now, we claim that the solution $u(x, t)$ also preserves the L_2 norm of the initial profile. Since u is an L_2 function on \mathbb{R} , by necessity u and all its derivatives vanish at $\pm\infty$. We take the Airy equation, multiply by u and integrate over the real line,

$$\int_{\mathbb{R}} u \cdot u_t dx = - \int_{\mathbb{R}} u \cdot u_{xxx} dx.$$

Recall, $(u^2)_t = 2u \cdot u_t$, and we integrate by parts on the right hand side, taking note that the integrated term vanishes,

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} \frac{1}{2} u^2 dx = \int_{\mathbb{R}} u_x \cdot u_{xx} dx.$$

Thus,

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} \frac{1}{2} u^2 dx = \int_{\mathbb{R}} u_x d(u_x) = \frac{(u_x)^2}{2} \Big|_{-\infty}^{\infty} = 0.$$

That is, $\int_{\mathbb{R}} u^2 dx$ is a constant in time, and hence that quantity is preserved.

Thus, $\|u\|_2 = \|F\|_2$, as desired.

3.4 Example Profiles

As we did for the Heat Equation, we now compute the solutions to the Airy Problem for the basic functions of interest. Because of the increased difficulty inherent with the Airy function, integrals may not be computed analytically, or show any significant simplification as they had for the heat problem. Instead, the solutions $u(x, t)$ are computed numerically.

3.4.1 Box Function

For the box shape initial profile

$$F(x) = \begin{cases} h & \text{if } -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

the solution is given as

$$u(x, t) = \frac{h}{\sqrt[3]{3t}} \int_{-a}^a \text{Ai} \left(\frac{x - \xi}{\sqrt[3]{3t}} \right) d\xi.$$

For simplicity, we take $a = h = 1$, and compute this integral numerically when plotting the solution u . For the values $t = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$, we plot u below in Figure 3.2.

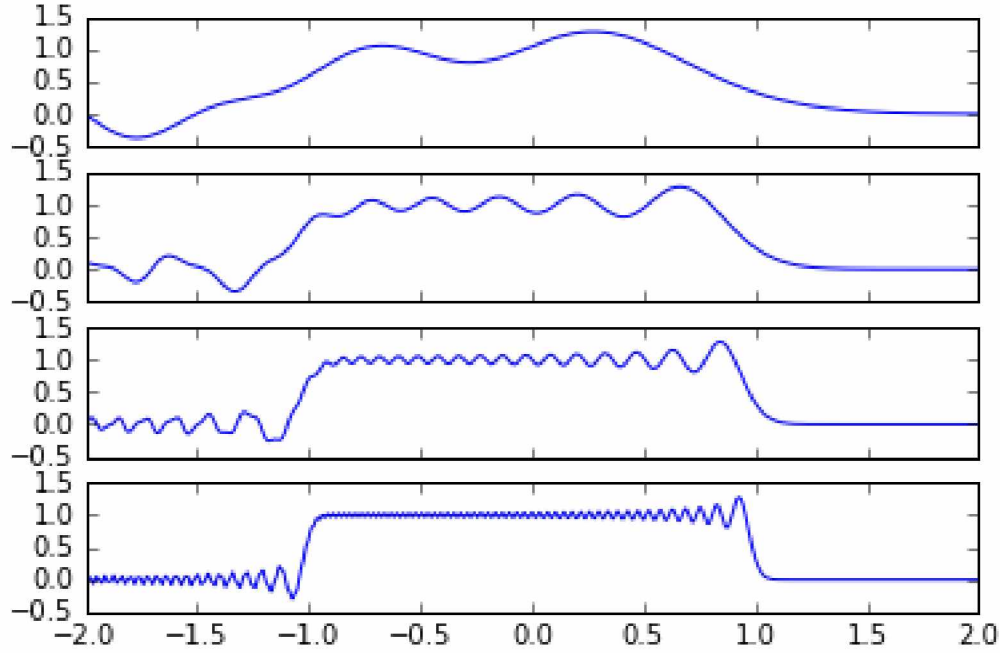


Figure 3.2: Solution to the Airy problem for box shape initial profile.

Note one significant difference from the solution to the heat problem IVP; since the Airy equation (3.1) and the Airy function (3.2) are not symmetric in x , the approximating function u is not symmetric in x . Also, near the point of discontinuity, we get significant oscillations of the approximating function above the height of the initial profile. This Gibbs phenomenon-like behavior was not observed for the solution to the heat equation.

The broken symmetry can be remedied somewhat by averaging the solution $u(x, t)$ with its reflection $u(-x, t)$,

$$u_{\text{symm}}(x, t) = \frac{1}{2} [u(x, t) + u(-x, t)].$$

The plot of this symmetrized function is plotted in Figure 3.3 for the previous values of t .

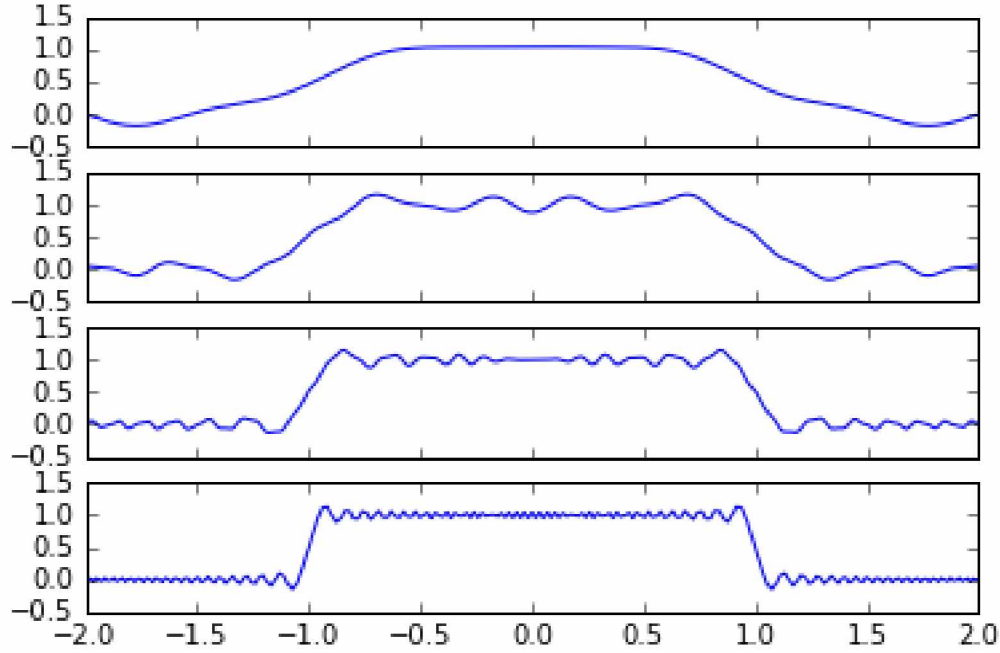


Figure 3.3: Symmetrized form of the solution to the Airy problem for box shape initial profile.

3.4.2 Tent Function

Now we consider the triangular shaped tent function defined by

$$F(x) = \begin{cases} \frac{h}{a}x + h & \text{if } -a \leq x \leq 0 \\ -\frac{h}{a}x + h & \text{if } 0 < x \leq a \\ 0 & \text{otherwise} \end{cases}$$

which yields a solution

$$u(x, t) = \frac{1}{\sqrt[3]{3t}} \int_{-a}^0 \left(\frac{h}{a}\xi + h \right) \text{Ai} \left(\frac{x - \xi}{\sqrt[3]{3t}} \right) d\xi + \frac{1}{\sqrt[3]{3t}} \int_0^a \left(-\frac{h}{a}\xi + h \right) \text{Ai} \left(\frac{x - \xi}{\sqrt[3]{3t}} \right) d\xi.$$

With $a = h = 1$, these integrals are computed numerically to find u for $t = 10^{-2}, 10^{-3}, 10^{-4}$, and 10^{-5} . The results are shown in Figure 3.4.

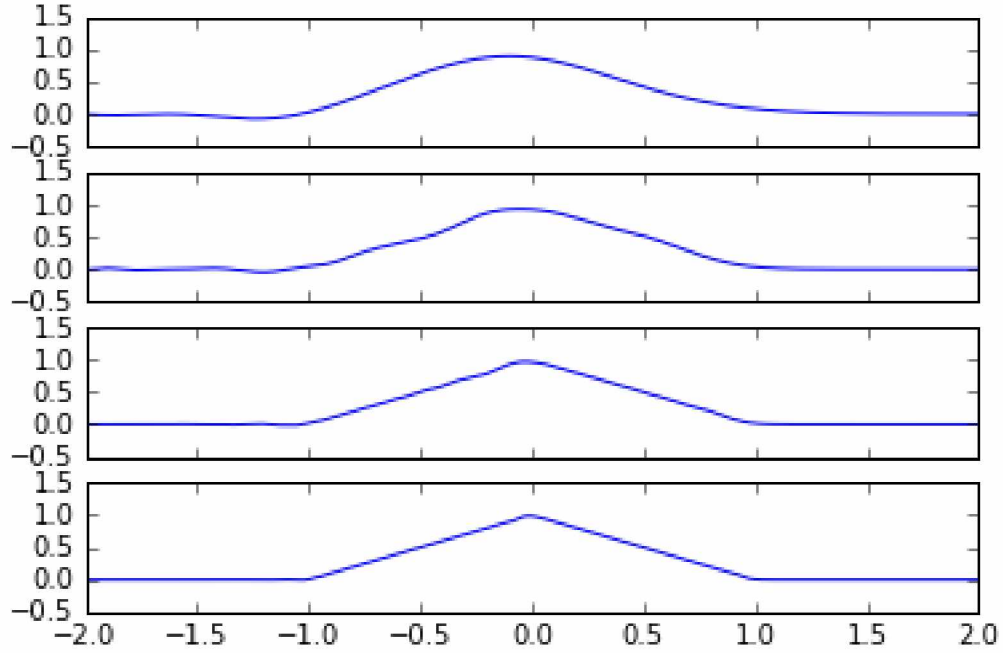


Figure 3.4: Solution to Airy problem for tent shape initial profile.

3.4.3 Cusp Function

The cusp shaped profile defined by

$$F(x) = e^{-\alpha|x|},$$

gives a solution of the form

$$u(x, t) = \frac{h}{\sqrt[3]{3t}} \int_{-\infty}^{\infty} e^{-\alpha|\xi|} \text{Ai} \left(\frac{x - \xi}{\sqrt[3]{3t}} \right) d\xi.$$

This integration is done numerically, with the constant $a = 1$, for values $t = .01, .001, .0001$, and $.00005$. The solution is plotted below in Figure 3.5.

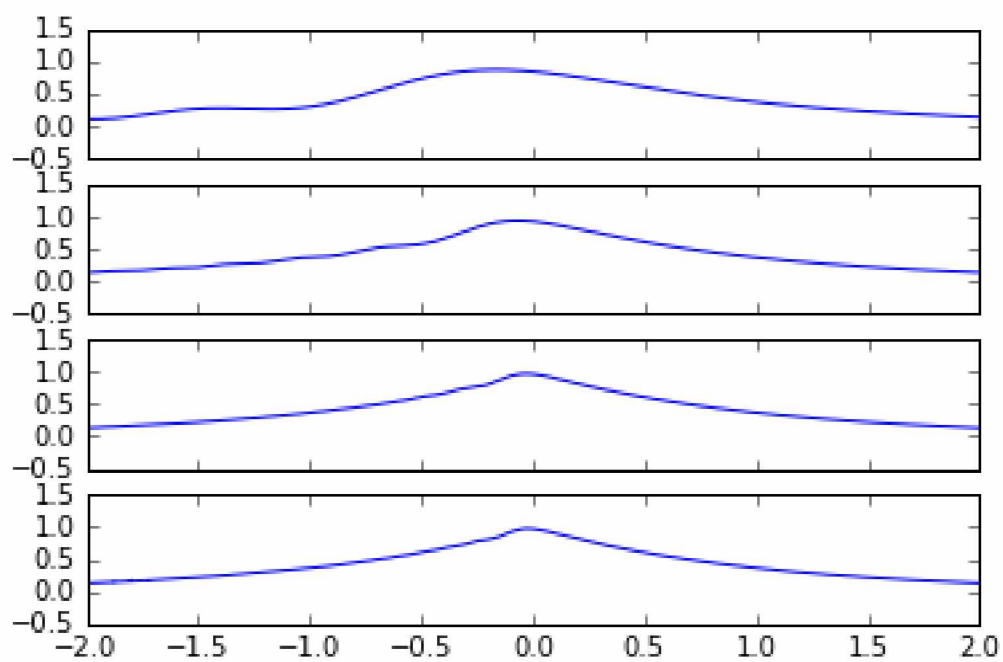


Figure 3.5: Solution to Airy problem for a cusp shape initial profile.

Chapter 4

Hermite Polynomials

4.1 Introduction

Consider the Hermite polynomials $H_n(x)$, which are a prominent polynomial sequence arising from such fields as statistics, numerical analysis, and as solutions to the quantum harmonic oscillator. They can be found by the generating function [But68] [Bay06]

$$G(x, t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n, \quad (4.1)$$

or by the Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \quad (4.2)$$

Explicitly, the first few Hermite polynomials are

$$\begin{aligned} H_0(x) &= 1 & H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 & H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 & H_5(x) &= 32x^5 - 160x^3 + 120x \end{aligned}$$

These polynomials are clearly not square integrable on the real line, so they are weighted by $e^{-x^2/2}$ so that they decay sufficiently at $\pm\infty$. We define

$$\Psi_n(x) = e^{-x^2/2} H_n(x)$$

and refer to these as weighted Hermite polynomials.

Lemma 6. *The weighted Hermite polynomials $\Psi_n(x) = e^{-x^2/2} H_n(x)$ are orthogonal,*

$$\int_{-\infty}^{\infty} \Psi_n(x) \Psi_m(x) dx = \begin{cases} 0 & \text{if } n \neq m \\ 2^n n! \sqrt{\pi} & \text{if } n = m. \end{cases}$$

Proof. Let n and m be integers, and without loss of generality, assume $n \geq m$. First, note that if we differentiate the generating function (4.1),

$$\frac{dG}{dx} = 2te^{-t^2+2xt} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{dH_n}{dx} t^n,$$

then rearrange terms to yield

$$e^{-t^2+2xt} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \frac{1}{2n} \frac{dH_n}{dx} t^{n-1}.$$

That is,

$$\frac{dH_n}{dx} = 2nH_{n-1} \quad (4.3)$$

and continuing this process yields

$$\frac{d^n H_n}{dx^n} = 2^n n! \quad (4.4)$$

since $H_0 = 1$.

We note that by the Rodrigues formula (4.2), $e^{-x^2} H_n(x) = (-1)^n \frac{d^n}{dx^n} e^{-x^2}$, and therefore

$$\begin{aligned} \int_{-\infty}^{\infty} \Psi_n \Psi_m dx &= \int_{-\infty}^{\infty} e^{-x^2} H_n H_m dx \\ &= \int_{-\infty}^{\infty} \left[(-1)^n \frac{d^n}{dx^n} e^{-x^2} \right] H_m dx. \end{aligned}$$

Integration by parts yields

$$\begin{aligned} (-1)^n \int_{-\infty}^{\infty} \left[\frac{d^n}{dx^n} e^{-x^2} \right] H_m dx &= \\ &= (-1)^n H_m \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \Big|_{-\infty}^{\infty} - (-1)^n \int_{-\infty}^{\infty} \frac{d^{n-1}}{dx^{n-1}} \left[e^{-x^2} \right] \frac{d}{dx} [H_m] dx \\ &= \int_{-\infty}^{\infty} e^{-x^2} H_{n-1} \frac{d}{dx} [H_m] dx, \end{aligned}$$

where we've used the fact that

$$(-1)^n \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} = -H_{n-1} e^{-x^2}$$

by the Rodrigues formula. Clearly, the integrated term vanishes at $\pm\infty$. Repeated integration by parts yields

$$\int_{-\infty}^{\infty} \Psi_n \Psi_m dx = \int_{-\infty}^{\infty} e^{-x^2} H_0 \frac{d^n}{dx^n} H_m dx,$$

where $H_0 = 1$. If $n > m$, the n -th derivative of H_m vanishes since it is an order m polynomial. If $n = m$, then by Equation (4.4),

$$\int_{-\infty}^{\infty} \Psi_n \Psi_n dx = \int_{-\infty}^{\infty} e^{-x^2} \frac{d^n}{dx^n} H_n dx = 2^n n! \int_{-\infty}^{\infty} e^{-x^2} dx = 2^n n! \sqrt{\pi},$$

and the result is established. \square

Now that we have the $L^2(\mathbb{R})$ norm established, we can define the normalized weighted Hermite polynomials

$$\psi_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x).$$

It is clear that $\{\psi_n\}$ is an orthonormal sequence. It can also be established [Joh14] that it is complete in $L^2(\mathbb{R})$, and hence forms a basis in this space. Thus, if the initial profile F is an $L^2(\mathbb{R})$ function, it can be expressed in terms of this basis,

$$F(x) = \sum_{k=0}^{\infty} c_k \psi_k(x), \quad \text{where } c_k = \int_{-\infty}^{\infty} F(\xi) \psi_k(\xi) d\xi.$$

Now, we define our approximating function as the *finite* sum,

$$u_n(x) = \sum_{k=0}^n c_k \psi_k(x).$$

Clearly, for each natural number n , u_n is a smooth $L^2(\mathbb{R})$ function since it is a finite linear combination of such functions.

4.2 Convergence

The most important feature of u_n is that it is a good approximation for the initial profile F . Naturally, this function converges in the L^2 norm, as we can see if we first note

$$\begin{aligned}
\|F(x) - u_n(x)\|^2 &= \left\| \sum_{k=n+1}^{\infty} c_k \psi_k \right\|^2 \\
&= \lim_{M \rightarrow \infty} \left\| \sum_{k=n+1}^M c_k \psi_k \right\|^2 \\
&= \lim_{M \rightarrow \infty} \left\langle \sum_{k=n+1}^M c_k \psi_k, \sum_{k=n+1}^M c_k \psi_k \right\rangle \\
&= \lim_{M \rightarrow \infty} \sum_{k=n+1}^M c_k^2 \\
&= \sum_{k=n+1}^{\infty} c_k^2,
\end{aligned}$$

where we've taken advantage of the fact that the functions ψ_k are orthonormal. Now, we know that the entire series converges, since F is in $L^2(\mathbb{R})$ by assumption,

$$\sum_{k=0}^{\infty} c_k^2 = \|F(x)\|^2 < \infty.$$

Therefore, the tail of this series must vanish, and hence

$$\lim_{n \rightarrow \infty} \|F(x) - u_n(x)\|^2 = \lim_{n \rightarrow \infty} \sum_{k=n+1}^{\infty} c_k^2 = 0,$$

and thus u_n converges to F in the L^2 norm.

4.3 Conserved Quantities

It appears that u_n does not conserve total area or energy. Indeed, the total area for the approximation u_n is

$$\int_{-\infty}^{\infty} u_n(x) dx = \int_{-\infty}^{\infty} \sum_{k=0}^n c_k \psi_k(x) dx = \sum_{k=0}^n c_k.$$

If we formally interchange the limit and integrals, the total area for the initial profile F is

$$\int_{-\infty}^{\infty} F(x) dx = \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} \sum_{k=0}^n c_k \psi_k(x) dx = \lim_{n \rightarrow \infty} \sum_{k=0}^n \int_{-\infty}^{\infty} c_k \psi_k(x) dx = \sum_{k=0}^{\infty} c_k.$$

Similarly, the energy is given by the integral of the square,

$$\int_{-\infty}^{\infty} [u_n(x)]^2 dx = \langle u_n, u_n \rangle^2 = \sum_{k=0}^n c_k^2,$$

and for the initial profile,

$$\int_{-\infty}^{\infty} [F(x)]^2 dx = \langle u_n, u_n \rangle^2 = \sum_{k=0}^{\infty} c_k^2.$$

For both of these quantities, the difference amounts to the tail of the series, which vanishes in the limit as argued above. But for a finite n , this difference may be significant.

4.4 Example Profiles

4.4.1 Box Function

As before, we consider the initial profile of a box function

$$F(x) = \begin{cases} h & \text{if } -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

with $a = h = 1$ for simplicity. Then, the approximation to this profile by Hermite polynomial expansion is

$$u_n(x) = \sum_{k=0}^n c_k \psi_k(x), \text{ where } c_k = \int_{-1}^1 \psi_k(x) dx.$$

Note that the Hermite polynomials are even functions for even index, and odd for odd index, as can be verified by the generating function (4.1) or by the Rodrigues formula (4.2). The normalized weighted Hermite polynomials also preserve this parity,

$$\psi_k(-x) = (-1)^k \psi_k(x).$$

Therefore, the coefficients c_k vanish for odd k since we are integrating over an even interval. These coefficients are found by integrating numerically before the functions are evaluated and partial sums are computed. The approximation $u_n(x)$ is shown below in Figure 4.1 for $n = 50, 100, 200$, and 250 terms.

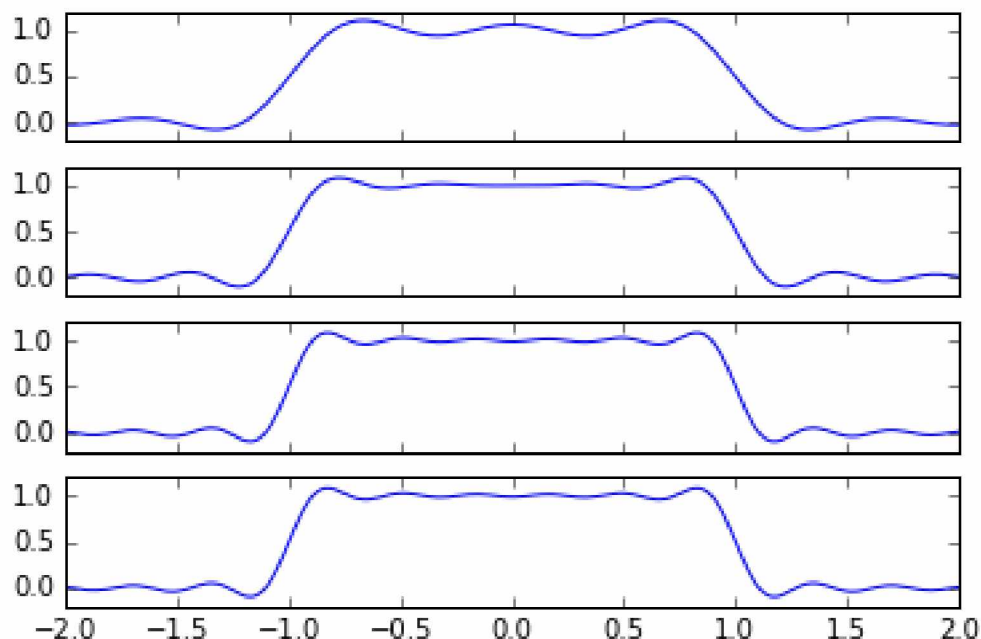


Figure 4.1: Solution for box profile by finite sums of Hermite polynomials.

4.4.2 Tent Function

Recall, the tent profile

$$F(x) = \begin{cases} \frac{h}{a}x + h & \text{if } -a \leq x \leq 0 \\ -\frac{h}{a}x + h & \text{if } 0 < x \leq a \\ 0 & \text{otherwise} \end{cases}$$

where we take $a = h = 1$ for simplicity in our calculations. As for the box function, the coefficients vanish for odd k when integrating over the even interval $[-1, 1]$. For even k , the coefficients are given by

$$c_k = \int_0^1 (1-x)\psi_k(x) dx,$$

which is integrated numerically before computing the partial sums $u_n(x)$. The plot below in Figure 4.2 shows the approximating function $u_n(x)$ for $n = 50, 100, 200$, and 250 terms.

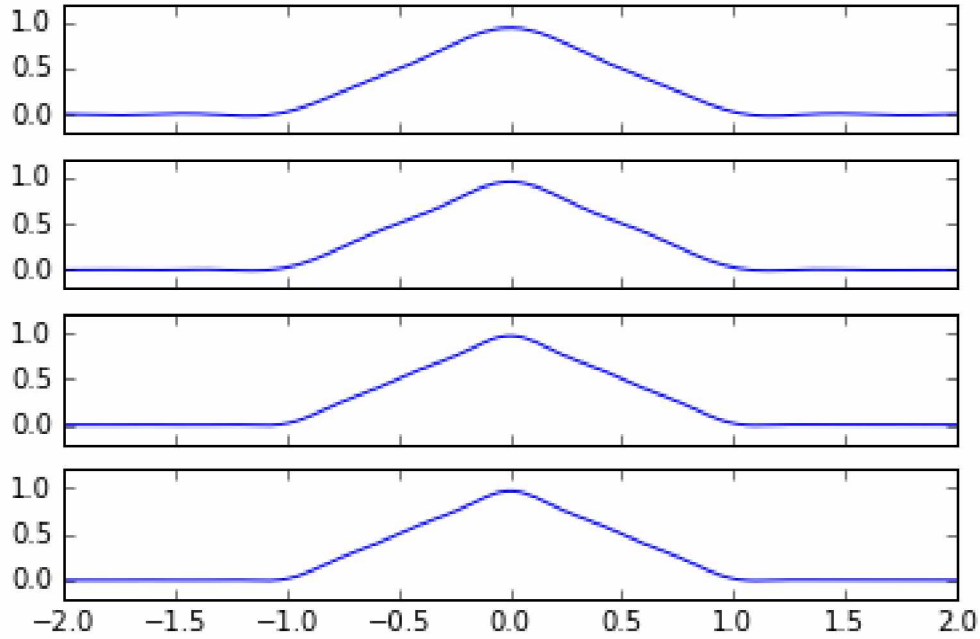


Figure 4.2: Solution for tent profile by finite sums of Hermite polynomials.

4.4.3 Cusp Function

As noted before, we consider the cusp function

$$F(x) = e^{-\alpha|x|},$$

with $\alpha = 1$ for simplicity. For practical purposes, when the coefficients

$$c_k = \int_{-\infty}^{\infty} e^{-|x|} \psi_k(x) dx$$

are calculated numerically, we use -10 to 10 as the limits of integration to avoid excessive roundoff error in the built-in integrator. As we did previously, the functions are evaluated and the partial sums computed for $n = 50, 100, 200$, and 250 terms. This is shown below in Figure 4.3.

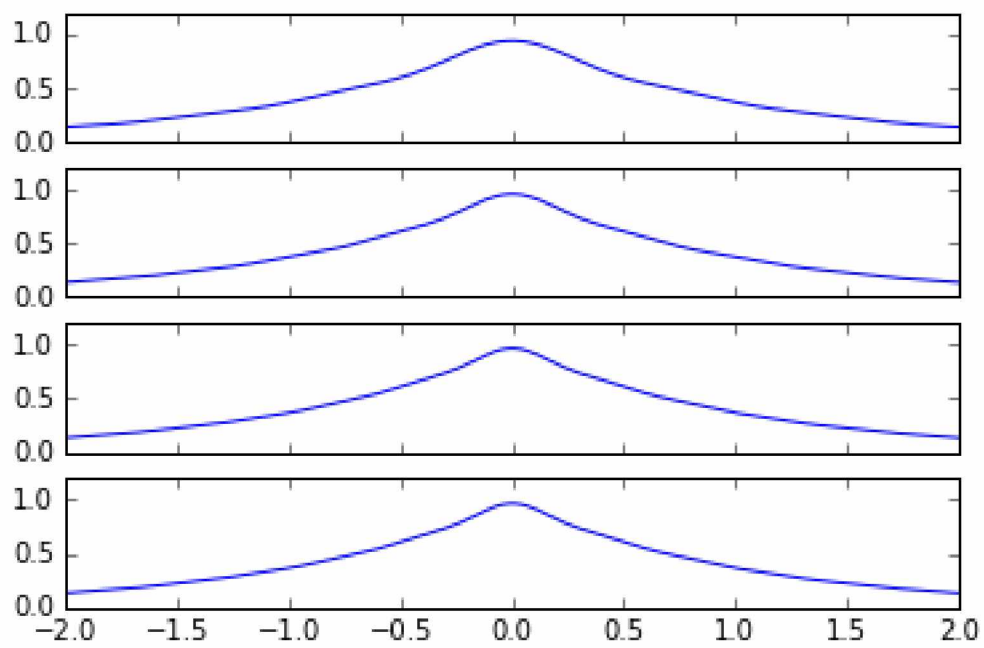


Figure 4.3: Solution for cusp profile by finite sums of Hermite polynomials.

Chapter 5

Conclusion

We have demonstrated two methods of approximating a potentially non-smooth L_2 function F on the line which yield several helpful properties. The solution $u(x, t)$ to the heat equation IVP with $u_{t=0} = F$ as the initial profile yields an approximation which extends to the complex plane, is an entire function, and hence smooth on the real line. This approximation converges weakly to the original function as $t \rightarrow 0$, and preserves the total area $\int_{\mathbb{R}} F dx = \int_{\mathbb{R}} u dx$ for all $t > 0$. Similarly, the solution $u(x, t)$ to the Airy equation IVP yields an approximation which again extends to the complex plane, is an entire function, and hence smooth on the real line. The solution by the Airy equation again preserves the total area of the original function, but also preserves the square integral, $\int_{\mathbb{R}} F^2 dx = \int_{\mathbb{R}} u^2 dx$, and hence the L_2 norm for all $t > 0$. For comparison, the weighted Hermite polynomials yield a smooth approximation which converges in the L_2 norm, but we get *no* conservation laws like we do for the PDE methods.

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